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Three-dimensional vibration analysis of thick rectangular plates using Chebyshev polynomial and Ritz method

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Abstract

This paper describes a method for free vibration analysis of rectangular plates with any thicknesses, which range from thin, moderately thick to very thick plates. It utilises admissible functions comprising the Chebyshev polynomials multiplied by a boundary function. The analysis is based on a linear, small-strain, three-dimensional elasticity theory. The proposed technique yields very accurate natural frequencies and mode shapes of rectangular plates with arbitrary boundary conditions. A very simple and general programme has been compiled for the purpose. For a plate with geometric symmetry, the vibration modes can be classified into symmetric and antisymmetric ones in that direction. In such a case, the computational cost can be greatly reduced while maintaining the same level of accuracy. Convergence studies and comparison have been carried out taking square plates with four simply-supported edges as examples. It is shown that the present method enables rapid convergence, stable numerical operation and very high computational accuracy. Parametric investigations on the vibration behaviour of rectangular plates with four clamped edges have also been performed in detail, with respect to different thickness-side ratios, aspect ratios and Poisson's ratios. These results may serve as benchmark solutions for validating approximate two-dimensional theories and new computational techniques in future.

Keywords: Three-dimensional vibration; Thick rectangular plate; Ritz method; Chebyshev polynomial; Natural frequencies

1. Introduction

Rectangular plates are commonly used structural components in aerospace, mechanical, nuclear, marine, and structural engineering. In some cases, the plates have to carry dynamic loads, and therefore a thorough understanding of their vibration characteristics is essential especially for designers. Despite the

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practical importance of elastic vibration solution of thick plate structures, closed-form, three-dimensional (3-D) elasticity solutions are limited only to rectangular plates with four simply-supported edges (Srinivas et al., 1970; Wittrick, 1987). In most cases, approximate numerical methods and/or analytical models have to be used.

A close scrutiny of the references on dynamic analysis of rectangular plates reveals that to date, most investigations were carried out based on two-dimensional theories, such as the classical plate theory (CPT) (Leissa, 1973), the first-order shear deformable plate theory (FSDPT) (Mindlin et al., 1956; Cheung and Zhou, 2000), and the high-order shear deformable plate theory (HSDPT) (Lim et al., 1998a,b). The two-dimensional theories reduce the dimensions of the plate problem from three to two by making certain hypotheses on the stress and strain in the thickness direction. These assumptions greatly simplify the formulation and solution in both analytical and computational methods, but they also introduce errors at the same time. For example, simplifications such as neglecting the transverse normal stress (Lim, 1999) are inherently erroneous. Since 3-D vibration analysis on the basis of linear, small-strain elasticity theory does not rely on any hypotheses involving the kinematics of deformation, such analysis not only provides realistic results but also brings out physical insights, which cannot otherwise be predicted by the two-dimensional analysis. Attempts have been made for 3-D vibration analysis of rectangular plates with general boundary conditions in the recent three decades. Cheung and Chakrabarti (1972) used the finite layer method to study the vibration of thick rectangular plates with general boundary conditions. Hutchinson and Zillimer (1983) and Fromme and Leissa (1970) used the series solution method to analyse the free vibration of a completely free parallelepiped. Malik and Bert (1998) and Liew and Teo (1999) used the differential quadrature (DQ) method to analyse the vibration characteristics of rectangular plates. Leissa and Zhang (1983) used simple algebraic polynomials and Liew et al. (1993, 1994, 1995) used orthogonal polynomials as admissible functions in the Ritz method to analyse such plates. In such work, the Ritz method shows some special advantages such as high accuracy, small computational cost and easy coding. However, the improvement in efficiency depends greatly on the choice of admissible functions. Various authors have reported on applications of the Ritz method to 3-D vibration analysis of structural components of other shapes (Lim et al., 1998c; Liew et al., 1998; Leissa and Kang, 1999; Leissa and So, 1995; So and Leissa, 1998) using simple algebraic polynomials and orthogonally generated polynomials as admissible functions. Recently, the authors (Cheung and Zhou, 2002; Zhou et al., 2002) analysed the 3-D vibration of triangular plates and tori with circular cross-section by using Chebyshev polynomials as the admissible functions in the Ritz method. High accuracy, stable computation and rapid convergence have been shown.

In this paper, Chebyshev polynomials multiplied by a boundary function are chosen to be the admissible functions for the analysis of 3-D vibration of rectangular plates of any thickness. The boundary function is chosen to satisfy the essential geometric boundary conditions of the plate, but it takes no account of the stress boundary conditions. For plates with geometric symmetry, the vibration modes can be further classified as symmetric and antisymmetric ones in that particular direction. In such a case, each of the categories can be separately determined and thus it results in a smaller set of eigenvalue equations while maintaining the same level of accuracy. Square plates with four simply-supported edges are taken as examples to show the convergence and accuracy of the present method. Rectangular plates with four clamped edges are then studied in detail and the numerical results are given in a tabulated form.

2. Mathematical formulation

The geometric configuration of a homogeneous isotropic rectangular plate is shown in Fig. 1. The plate has a length a, a width b and a uniform thickness t. The plate geometry and dimensions are defined with respect to a Cartesian coordinate system (x, y, z), the origin of which is at the centre of the plate and the axes

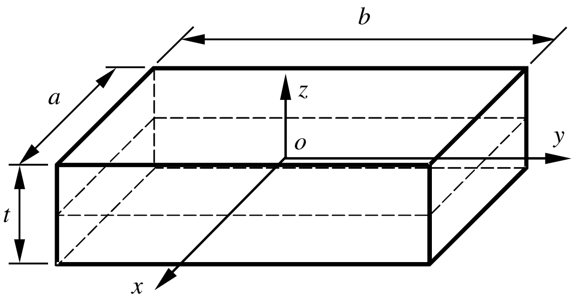

Fig. 1. Geometry, dimensions and coordinates of a rectangular plate with uniform thickness.

are parallel to the edges of the plate. The corresponding displacement components at a generic point are u, v and w in the x, y and z directions, respectively.

The linear elastic strain energy V for a rectangular plate can be written in integral form as

$$V = \frac{E}{2(1+v)} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-t/2}^{t/2} \left(\frac{v\Lambda_1^2}{1-2v} + \Lambda_2 + \frac{\Lambda_3}{2} \right) \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \tag{1}$$

where

$$\Lambda_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}; \quad \Lambda_2 = \varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2; \quad \Lambda_3 = \varepsilon_{xy}^2 + \varepsilon_{xz}^2 + \varepsilon_{yz}^2 \tag{2}$$

E is the Young's modulus and v is the Poisson' ratio. $\varepsilon_{ij}(i,j = x,y,z)$ are the strain components in the Cartesian coordinates for small deformation, which are given as

$$\begin{aligned} &\varepsilon_{xx} = \frac{\partial u}{\partial x}; \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}; \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \\ &\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \quad \varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \quad \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{aligned} \tag{3}$$

The kinetic energy T of the plate can be written as

$$T = \frac{\rho}{2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-t/2}^{t/2} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \tag{4}$$

where ρ is the mass density per unit volume.

For a plate undergoing free vibration, its periodic displacement components can be expressed in terms of the displacement amplitude functions as follows

$$u(x,y,z,t) = U(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}; \quad v(x,y,z,t) = V(x,y,z)\mathrm{e}^{\mathrm{i}\omega t}; \quad w(x,y,z,t) = W(x,y,z)\mathrm{e}^{\mathrm{i}\omega t} \tag{5}$$

where ω denotes the natural frequency of the plate and $\mathrm{i} = \sqrt{-1}$.

For simplicity and convenience in mathematical formulation, the following non-dimensional parameters are introduced

$$\xi = 2x/a; \quad \eta = 2y/b; \quad \zeta = 2z/t \tag{6}$$

The maximum energy functional Π of the plate is defined as

$$\Pi = V_{\max} - T_{\max} \tag{7}$$

where

$$V_{\max} = \frac{Et}{4\lambda(1+v)} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \left(\frac{v\overline{\Lambda}_1^2}{1-2v} + \overline{\Lambda}_2 + \frac{\overline{\Lambda}_3}{2} \right) \mathrm{d}\zeta\, \mathrm{d}\eta\, \mathrm{d}\xi;$$
$$T_{\max} = \frac{\rho}{16} abt\omega^2 \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (U^2 + V^2 + W^2)\, \mathrm{d}\zeta\, \mathrm{d}\eta\, \mathrm{d}\xi \tag{8}$$

in which

$$\overline{\Lambda}_1 = \bar{\varepsilon}_{\xi\xi} + \bar{\varepsilon}_{\eta\eta} + \bar{\varepsilon}_{\zeta\zeta}; \quad \overline{\Lambda}_2 = \bar{\varepsilon}_{\xi\xi}^2 + \bar{\varepsilon}_{\eta\eta}^2 + \bar{\varepsilon}_{\zeta\zeta}^2; \quad \overline{\Lambda}_3 = \bar{\varepsilon}_{\xi\eta}^2 + \bar{\varepsilon}_{\xi\zeta}^2 + \bar{\varepsilon}_{\eta\zeta}^2,$$
$$\bar{\varepsilon}_{\xi\xi} = \frac{\partial U}{\partial \xi}; \quad \bar{\varepsilon}_{\eta\eta} = \lambda \frac{\partial V}{\partial \eta}; \quad \bar{\varepsilon}_{\zeta\zeta} = \frac{\lambda}{\gamma} \frac{\partial W}{\partial \zeta},$$
$$\bar{\varepsilon}_{\xi\eta} = \lambda \frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi}; \quad \bar{\varepsilon}_{\xi\zeta} = \frac{\lambda}{\gamma} \frac{\partial U}{\partial \zeta} + \frac{\partial W}{\partial \xi}; \quad \bar{\varepsilon}_{\eta\zeta} = \frac{\lambda}{\gamma} \frac{\partial V}{\partial \zeta} + \lambda \frac{\partial W}{\partial \eta}, \tag{9}$$
$$\lambda = a/b; \quad \gamma = t/b$$

Each of the displacement amplitude functions $U(\xi, \eta, \zeta)$; $V(\xi, \eta, \zeta)$ and $W(\xi, \eta, \zeta)$ is written, respectively, in the form of triplicate series of Chebyshev polynomials multiplied by a boundary function which ensures that the displacement component satisfies the essential geometric boundary conditions of the plate, i.e.

$$U(\xi, \eta, \zeta) = F_u(\xi, \eta) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{ijk} P_i(\xi) P_j(\eta) P_k(\zeta);$$
$$V(\xi, \eta, \zeta) = F_v(\xi, \eta) \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{lmn} P_l(\xi) P_m(\eta) P_n(\zeta); \tag{10}$$
$$W(\xi, \eta, \zeta) = F_w(\xi, \eta) \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{r=1}^{\infty} C_{pqr} P_p(\xi) P_q(\eta) P_r(\zeta)$$

where $P_s(\chi)$ $(s = 1, 2, 3, \ldots.; \chi = \xi, \eta, \zeta)$ is the one-dimensional sth Chebyshev polynomial which can be written in terms of cosine functions as follows

$$P_s(\chi) = \cos[(s-1)\arccos(\chi)]; \quad (s = 1, 2, 3, \ldots) \tag{11}$$

Note that $F_u(\xi, \eta)$; $F_v(\xi, \eta)$ and $F_w(\xi, \eta)$ are the boundary functions, respectively, corresponding to the displacement amplitude functions $U(\xi, \eta, \zeta)$; $V(\xi, \eta, \zeta)$ and $W(\xi, \eta, \zeta)$, which can be written as

$$F_\delta(\xi, \eta) = f_\delta^1(\xi) f_\delta^2(\eta); \quad (\delta = u, v, w) \tag{12}$$

The boundary function components f_δ^i $(\delta = u; v; w, i = 1; 2)$ corresponding to different boundary conditions are given in Table 1.

It should be noted that using the Chebyshev polynomial series as the admissible functions has two distinct advantages (Fox and Parker, 1968). One is that $P_s(\chi)$ $(s = 1, 2, 3, \ldots)$ is a set of complete and orthogonal series in the interval $[-1, 1]$. This ensures that the three duplicate series $P_i(\xi)P_j(\eta)P_k(\zeta)$ $(i, j, k, = 1, 2, 3, \ldots)$ is a complete and orthogonal set in the plate region. Therefore, more rapid convergence and better stability in the numerical computation can be accomplished compared with other polynomial series. The other is that the Chebyshev polynomial and its derivatives can be expressed in simple and uniform form, which reduces the coding effort. The first five terms of the Chebyshev polynomials are shown in Fig. 2.

Table 1
Boundary functions for different boundary conditions

Boundary condition	$f_u^1(\xi)$	$f_v^1(\xi)$	$f_w^1(\xi)$	$f_u^2(\eta)$	$f_v^2(\eta)$	$f_w^2(\eta)$
F-F	1	1	1	1	1	1
F-S	1	$1-\xi$	$1-\xi$	$1-\eta$	1	$1-\eta$
S-F	1	$1+\xi$	$1+\xi$	$1+\eta$	1	$1+\eta$
S-S	1	$1-\xi^2$	$1-\xi^2$	$1-\eta^2$	1	$1-\eta^2$
F-C	$1-\xi$	$1-\xi$	$1-\xi$	$1-\eta$	$1-\eta$	$1-\eta$
C-F	$1+\xi$	$1+\xi$	$1+\xi$	$1+\eta$	$1+\eta$	$1+\eta$
S-C	$1-\xi$	$1-\xi^2$	$1-\xi^2$	$1-\eta^2$	$1-\eta$	$1-\eta^2$
C-S	$1+\xi$	$1-\xi^2$	$1-\xi^2$	$1-\eta^2$	$1+\eta$	$1-\eta^2$
C-C	$1-\xi^2$	$1-\xi^2$	$1-\xi^2$	$1-\eta^2$	$1-\eta^2$	$1-\eta^2$

Legend for boundary conditions: 1. The first letter refers to the edge at $\xi=-1$ or $\eta=-1$ whereas the second letter refers to that at $\xi=1$ or $\eta=1$. 2. F, free edge; S, simply-supported edge; C, clamped edge.

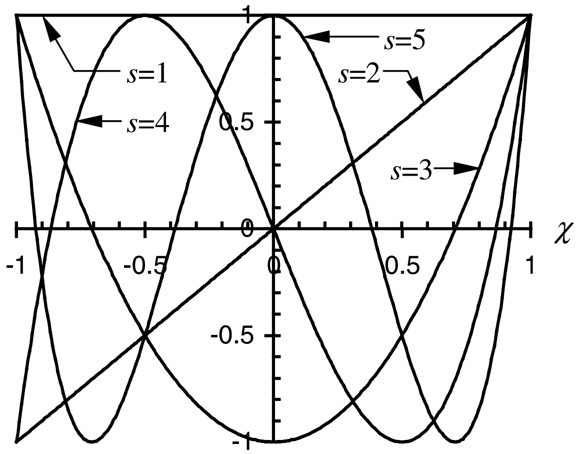

Fig. 2. The first five terms of the Chebyshev polynomials $F_s(\chi)(s=1,2,3,4,5)$.

Substituting Eq. (10) into Eq. (7) and minimizing the functional Π with respect to the coefficients of the admissible functions, i.e.

$$\frac{\partial \Pi}{\partial A_{ijk}}=0; \quad \frac{\partial \Pi}{\partial B_{lmn}}=0; \quad \frac{\partial \Pi}{\partial C_{pqr}}=0 \quad (i;j;k,l;m;n,p;q;r=1,2,3,\ldots) \tag{13}$$

leads to the following governing eigenvalue equation in matrix form

$$\left(\begin{bmatrix} [K_{uu}] & [K_{uv}] & [K_{uw}] \\ [K_{uv}]^{\mathrm{T}} & [K_{vv}] & [K_{vw}] \\ [K_{uw}]^{\mathrm{T}} & [K_{vw}]^{\mathrm{T}} & [K_{ww}] \end{bmatrix} - \Omega^2 \begin{bmatrix} [M_{uu}] & 0 & 0 \\ 0 & [M_{vv}] & 0 \\ 0 & 0 & [M_{ww}] \end{bmatrix}\right) \begin{Bmatrix} \{A\} \\ \{B\} \\ \{C\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix} \tag{14}$$

in which $\Omega=\omega a\sqrt{\rho/E}$, $[K_{ij}]$ and $[M_{ii}]$ $(i,j=u,v,w)$ are the stiffness sub-matrices and the diagonal mass sub-matrices, respectively. The column vectors $\{A\}$, $\{B\}$ and $\{C\}$ contain unknown coefficients expressed in the following forms

$$\{A\} = \begin{Bmatrix} A_{111} \\ A_{112} \\ \vdots \\ A_{11K} \\ A_{121} \\ \vdots \\ A_{12K} \\ \vdots \\ A_{1JK} \\ \vdots \\ A_{IJK} \end{Bmatrix}, \quad \{B\} = \begin{Bmatrix} B_{111} \\ B_{112} \\ \vdots \\ B_{11N} \\ B_{121} \\ \vdots \\ B_{12N} \\ \vdots \\ B_{1MN} \\ \vdots \\ A_{LMN} \end{Bmatrix}, \quad \{C\} = \begin{Bmatrix} C_{111} \\ C_{112} \\ \vdots \\ C_{11R} \\ C_{121} \\ \vdots \\ C_{12R} \\ \vdots \\ C_{1QR} \\ \vdots \\ C_{PQR} \end{Bmatrix} \tag{15}$$

The elements of the stiffness sub-matrices $[K_{ij}]$ and mass sub-matrices $[M_{ii}]$ $(i,j = u,v,w)$ are given by

$$\begin{aligned}
&[K_{uu}] = \frac{1-v}{1-2v} D^{1,1}_{uiu\bar{i}} G^{0,0}_{uju\bar{j}} H^{0,0}_{uku\bar{k}} + \frac{\lambda^2}{2}\left(D^{0,0}_{uiu\bar{i}} G^{1,1}_{uju\bar{j}} H^{0,0}_{uku\bar{k}} + \frac{1}{\gamma^2} D^{0,0}_{uiu\bar{i}} G^{0,0}_{uju\bar{j}} H^{1,1}_{uku\bar{k}} \right);\\
&[K_{uv}] = \lambda\left(\frac{v}{1-2v} D^{1,0}_{uivl} G^{0,1}_{ujuvm} H^{0,0}_{ukun} + \frac{1}{2} D^{0,1}_{uivl} G^{1,0}_{ujvm} H^{0,0}_{ukun} \right);\\
&[K_{uw}] = \frac{\lambda}{\gamma}\left(\frac{v}{1-2v} D^{1,0}_{uiwp} G^{0,0}_{ujuwq} H^{0,1}_{ukwr} + \frac{1}{2} D^{0,1}_{uiwp} G^{0,0}_{ujuwq} H^{1,0}_{ukwr} \right);\\
&[K_{vv}] = \lambda^2\left(\frac{1-v}{1-2v} D^{0,0}_{vlv\bar{l}} G^{1,1}_{vmv\bar{m}} H^{0,0}_{vnv\bar{n}} + \frac{1}{2\gamma^2} D^{0,0}_{vlv\bar{l}} G^{0,0}_{vmv\bar{m}} H^{1,1}_{vnv\bar{n}} \right) + \frac{1}{2} D^{1,1}_{vlv\bar{l}} G^{0,0}_{vmv\bar{m}} H^{0,0}_{vnv\bar{n}};\\
&[K_{vw}] = \frac{\lambda^2}{\gamma}\left(\frac{v}{1-2v} D^{0,0}_{vlwp} G^{1,0}_{vmwq} H^{0,1}_{vnwr} + \frac{1}{2} D^{0,0}_{vlwp} G^{0,1}_{vmwq} H^{1,0}_{vnwr} \right);\\
&[K_{ww}] = \lambda^2\left(\frac{1-v}{\gamma^2(1-2v)} D^{0,0}_{wpw\bar{p}} G^{0,0}_{wqw\bar{q}} H^{1,1}_{wrw\bar{r}} + \frac{1}{2} D^{0,0}_{wpw\bar{p}} G^{1,1}_{wqw\bar{q}} H^{0,0}_{wrw\bar{r}} \right) + \frac{1}{2} D^{1,1}_{wpw\bar{p}} G^{0,0}_{wqw\bar{q}} H^{0,0}_{wrw\bar{r}},\\
&[M_{uu}] = (1+v) D^{0,0}_{uiu\bar{i}} G^{0,0}_{uju\bar{j}} H^{0,0}_{uku\bar{k}}/4;\\
&[M_{vv}] = (1+v) D^{0,0}_{vlv\bar{l}} G^{0,0}_{vmv\bar{m}} H^{0,0}_{vnv\bar{n}}/4;\\
&[M_{ww}] = (1+v) D^{0,0}_{wpw\bar{p}} G^{0,0}_{wqw\bar{q}} H^{0,0}_{wrw\bar{r}}/4
\end{aligned} \tag{16}$$

in which

$$\begin{aligned}
&D^{s,\bar{s}}_{\alpha\sigma\beta\theta} = \int_{-1}^{1} \left\{ \frac{\mathrm{d}^s[f^1_\alpha(\xi)P_\sigma(\xi)]}{\mathrm{d}\xi^s} \frac{\mathrm{d}^{\bar{s}}[f^1_\beta(\xi)P_\theta(\xi)]}{\mathrm{d}\xi^{\bar{s}}} \right\} \mathrm{d}\xi;\\
&G^{s,\bar{s}}_{\alpha\sigma\beta\theta} = \int_{-1}^{1} \left\{ \frac{\mathrm{d}^s[f^2_\alpha(\eta)P_\sigma(\eta)]}{\mathrm{d}\eta^s} \frac{\mathrm{d}^{\bar{s}}[f^2_\beta(\eta)P_\theta(\eta)]}{\mathrm{d}\eta^{\bar{s}}} \right\} \mathrm{d}\eta;\\
&H^{s,\bar{s}}_{\alpha\sigma\beta\theta} = \int_{-1}^{1} \left\{ \frac{\mathrm{d}^s P_\sigma(\zeta)}{\mathrm{d}\zeta^s} \frac{\mathrm{d}^{\bar{s}} P_\theta(\zeta)}{\mathrm{d}\zeta^{\bar{s}}} \right\} \mathrm{d}\zeta;\\
&(s;\bar{s} = 0;1, \quad \alpha;\beta = u;v;w, \quad \sigma = i;j;k;l;m;n;p;q;r, \quad \theta = \bar{i};\bar{j};\bar{k};\bar{l};\bar{m};\bar{n};\bar{p};\bar{q};\bar{r})
\end{aligned} \tag{17}$$

Table 2
The Chebyshev polynomials corresponding to different mode categories

Geometric symmetry	Symmetric modes			Antisymmetric modes		
	U	V	W	U	V	W
x-direction	$i = 2, 4, 6, \dots$	$l = 1, 3, 5, \dots$	$p = 1, 3, 5, \dots$	$i = 1, 3, 5, \dots$	$l = 2, 4, 6, \dots$	$p = 2, 4, 6, \dots$
y-direction	$j = 1, 3, 5, \dots$	$m = 2, 4, 6, \dots$	$q = 1, 3, 5, \dots$	$j = 2, 4, 6, \dots$	$m = 1, 3, 5, \dots$	$q = 2, 4, 6, \dots$
z-direction	$k = 1, 3, 5, \dots$	$n = 1, 3, 5, \dots$	$r = 2, 4, 6, \dots$	$k = 2, 4, 6, \dots$	$n = 2, 4, 6, \dots$	$r = 1, 3, 5, \dots$

Solving the eigenvalue Eq. (14) yields the frequency parameters Ω. The mode shape corresponding to each eigenvalue may be obtained by back-substitution of the eigenvalues, one by one, in the usual manner.

It is well known that the Chebyshev polynomial $P_s(\chi)$ is symmetric for $s = 1, 3, 5, \dots$ and antisymmetric for $s = 2, 4, 6, \dots$. From Table 1, one can find that the boundary functions $f_\delta^1(\xi)$ ($\delta = u, v, w$) are symmetric for a plate with the same support conditions at $\xi = 1$ and $\xi = -1$. The boundary functions $f_\delta^2(\eta)$ ($\delta = u, v, w$) are symmetric for a plate with the same support conditions at $\eta = 1$ and $\eta = -1$. This means that for a plate with geometric symmetry, its vibration modes may be classified into symmetric modes and antisymmetric modes. These two separate categories of modes can be separately determined, resulting in a smaller set of eigenvalue equations while maintaining the same level of accuracy. First and foremost, the rectangular plates considered here have uniform thicknesses, which are invariably symmetric about the mid-plane. Therefore the vibration modes of a plate can be divided into at least two categories: symmetric modes (S) and antisymmetric modes (A) about the coordinate ζ. Furthermore if a plate has symmetric boundary supports only in the ξ (or the η) direction, then its vibration modes can be divided into four distinct categories: AA, AS, SA and SS, in which the first capital letter refers to the type of vibration modes in the ξ (or the η) direction and the second refers to that in the ζ-direction. Similarly if a plate has symmetric boundary supports in both ξ and η directions, then its vibration modes can be divided into eight distinct categories: AAA, AAS, ASA, ASS, SAA, SAS, SSA and SSS, in which the first, second and third capital letters refer, respectively, to the type of vibration modes in the ξ, η and ζ-directions. The Chebyshev polynomials corresponding to different mode categories are given in Table 2.

3. Convergence study and comparison

As it is well known, the Ritz method can provide accurate solutions. However, its efficiency depends greatly on the choice of global admissible functions. The natural frequencies obtained by the Ritz method converge as upper bounds to the exact values. These upper bound estimates could be improved by increasing the number of terms of admissible functions in the computation and hence solution of any accuracy can be obtained in theory. However, a practical limit to the number of terms used always exists because of the limited speed, the capacity and the numerical accuracy of computers. In the 3-D vibration analysis of an elastic body in particular, numerical instability may occur with a great number of terms of admissible functions, especially when triplicate series are used. Therefore, the validity of a numerical method often hinges upon the convergence rate, the numerical stability and the accuracy of the method.

Square plates (i.e. aspect ratio $\lambda = 1$) with four simply-supported edges are taken as an example for the convergence studies. Three different thickness-side ratios $\gamma = 0.01$; 0.2; 0.5 are considered, which correspond, respectively, to thin, moderately thick and very thick plates. The Poisson's ratio $\nu = 0.3$ is used in the study. All the computations were performed in double precision (16 significant figures) and piecewise Gaussian quadrature was used numerically to evaluate the integrals in Eq. (17). To facilitate comparison, the non-dimensional frequency parameters are expressed as

Table 3
Square plates with four simply-supported edges: convergence of frequency parameters Δ for the SSA modes

Terms in x, y, z	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
$t/b = 0.01$												
$8 \times 8 \times 1$	2.2128	11.048	11.048	19.856	28.640	28.640	37.397	37.397	54.836	54.839	54.839	63.521
$8 \times 8 \times 2$	1.9993	9.9826	9.9826	17.944	25.883	25.883	33.800	33.800	49.569	49.571	49.571	57.424
$9 \times 9 \times 1$	2.2128	11.048	11.048	19.856	28.640	28.640	37.397	37.397	54.836	54.836	54.836	63.519
$9 \times 9 \times 2$	1.9993	9.9826	9.9826	17.944	25.883	25.883	33.800	33.800	49.569	49.569	49.569	57.422
$10 \times 10 \times 3$	1.9993	9.9826	9.9826	17.944	25.883	25.883	33.800	33.800	49.569	49.569	49.569	57.422
$t/b = 0.2$												
$6 \times 6 \times 2$	1.7760	6.6959	6.6959	10.182	13.015	13.015	15.454	15.454	16.953	18.354	19.300	19.300
$6 \times 6 \times 3$	1.7758	6.6868	6.6868	10.153	12.957	12.957	15.361	15.361	16.948	18.324	19.297	19.297
$7 \times 7 \times 2$	1.7760	6.6959	6.6959	10.182	13.014	13.014	15.454	15.454	16.953	18.354	19.300	19.300
$7 \times 7 \times 3$	1.7758	6.6868	6.6868	10.153	12.956	12.956	15.361	15.361	16.948	18.324	19.297	19.297
$8 \times 8 \times 4$	1.7758	6.6868	6.6868	10.153	12.956	12.956	15.361	15.361	16.948	18.324	19.297	19.297
$t/b = 0.5$												
$6 \times 6 \times 4$	1.2590	3.1958	3.5431	3.5431	4.0770	4.8817	4.8817	4.9353	6.0230	6.0230	6.1195	6.5400
$6 \times 6 \times 5$	1.2590	3.1958	3.5431	3.5431	4.0770	4.8817	4.8817	4.9353	6.0230	6.0230	6.1195	6.5399
$7 \times 7 \times 4$	1.2590	3.1958	3.5431	3.5431	4.0770	4.8817	4.8817	4.9353	6.0229	6.0229	6.1195	6.5400
$7 \times 7 \times 5$	1.2590	3.1958	3.5431	3.5431	4.0770	4.8817	4.8817	4.9353	6.0228	6.0228	6.1195	6.5399
$8 \times 8 \times 6$	1.2590	3.1958	3.5431	3.5431	4.0770	4.8817	4.8817	4.9353	6.0228	6.0228	6.1195	6.5399

Table 4
Square plates with four simply-supported edges: convergence of freauency parameters Δ for the SSS modes

Terms in x, y, z	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
$t/b = 0.01$												
$7 \times 7 \times 1$	92.255	155.94	206.29	206.29	276.76	332.63	332.63	348.67	348.67	380.38	380.38	461.28
$7 \times 7 \times 2$	92.255	155.94	206.29	206.29	276.76	332.63	332.63	348.66	348.66	380.38	380.38	461.28
$8 \times 8 \times 1$	92.255	155.94	206.29	206.29	276.76	332.63	332.63	348.67	348.67	380.38	380.38	461.27
$8 \times 8 \times 2$	92.255	155.94	206.29	206.29	276.76	332.63	332.63	348.66	348.66	380.38	380.38	461.27
$9 \times 9 \times 3$	92.255	155.94	260.29	206.29	276.76	332.63	332.63	348.66	348.66	380.38	380.38	461.27
$t/b = 0.2$												
$6 \times 6 \times 2$	4.6127	7.7465	10.314	10.314	13.838	16.632	16.632	16.679	16.679	19.020	19.020	21.010
$6 \times 6 \times 3$	4.6127	7.7465	10.314	10.314	13.838	16.632	16.632	16.676	16.676	19.020	19.020	20.985
$7 \times 7 \times 2$	4.6127	7.7465	10.314	10.314	13.838	16.632	16.632	16.679	16.679	19.019	19.019	21.010
$7 \times 7 \times 3$	4.6127	7.7465	10.314	10.314	13.838	16.632	16.632	16.676	16.676	19.019	19.019	20.985
$8 \times 8 \times 4$	4.6127	7.7465	10.314	10.314	13.838	16.632	16.632	16.676	16.676	19.019	19.019	20.985
$t/b = 0.5$												
$6 \times 6 \times 4$	1.8451	2.9325	4.1258	4.1258	4.5772	4.5772	4.6142	5.5353	5.5354	5.5457	6.4278	6.4278
$6 \times 6 \times 5$	1.8451	2.9325	4.1258	4.1258	4.5772	4.5772	4.6142	5.5353	5.5353	5.5457	6.4278	6.4278
$7 \times 7 \times 4$	1.8451	2.9325	4.1258	4.1258	4.5772	4.5772	4.6142	5.5353	5.5354	5.5457	6.4275	6.4275
$7 \times 7 \times 5$	1.8451	2.9325	4.1258	4.1258	4.5772	4.5772	4.6142	5.5353	5.5353	5.5457	6.4275	6.4275
$8 \times 8 \times 6$	1.8451	2.9325	4.1258	4.1258	4.5772	4.5772	4.6142	5.5353	5.5353	5.5457	6.4275	6.4275

$$\Delta = \frac{b}{at\pi^2}\sqrt{12(1-\nu^2)}\Omega = \frac{\omega b^2}{\pi^2}\sqrt{\rho t/D} \tag{18}$$

where $D = Et^3/[12(1-\nu^2)]$ is the flexural rigidity of the plate.

Tables 3 and 4 give the rate of convergence of the first 12 frequency parameters Δ_i $(i = 1, 2, \ldots, 12)$ for the SSA and SSS modes up to five significant figures, respectively. It is seen that the convergence rate is very rapid. The frequency parameters monotonically decrease and approach the exact values with the increase in the number of terms of admissible functions. The convergence patterns are also found to be similar for all of the mode categories. In general, with the increase in plate thickness, more terms in the thickness direction are needed compared with the other directions. The minimum numbers of terms from which results of the first 12 frequency parameters for each mode category are accurate to five significant figures are summarized in Table 5.

The first eight frequency parameters for simply-supported square plates with four different thickness-side ratio $t/b = 0.001$, 0.1, 0.2, 0.5 computed using the present method are given in Table 6 and compared with other published solutions. Note that $t/b = 0.001$ corresponds to a very thin plate; $t/b = 0.1$ and $t/b = 0.2$ correspond to moderately thick plates whereas $t/b = 0.5$ corresponds to a very thick plate. Good agreement has been observed for all cases ranging from very thin to very thick cases. It should be pointed out that the present method possesses rather good stability in numerical computation. The case of a very thin square

Table 5
Square plates with four simply-supported edges: minimum number of terms required to give the first 12 frequency parameters convergent to five significant figures

t/b	Mode category					
	AAA	AAS	ASA (SAA)	ASS (SAS)	SSA	SSS
0.01	10 × 10 × 2	8 × 8 × 2	10 × 10 × 2	8 × 8 × 2	9 × 9 × 2	8 × 8 × 2
0.2	8 × 8 × 4	8 × 8 × 3	8 × 8 × 4	8 × 8 × 3	7 × 7 × 3	7 × 7 × 3
0.5	6 × 6 × 4	7 × 7 × 5	7 × 7 × 5	7 × 7 × 5	7 × 7 × 5	7 × 7 × 5

Table 6
Square plates with four simply-supported edges: comparison of the first eight frequency parameters Δ

t/b	Solution method	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
0.001	Classical theory (Leissa, 1973)	2.0	5.0	5.0	8.0	10.0	10.0	13.0	13.0
	Present 3-D solutions	1.9972	4.9999	4.9999	7.9996	9.9995	9.9995	13.000	13.000
0.1	Mindlin theory (Mindlin et al., 1956)	1.9311	4.6048	4.6048	–	–	7.0637	8.6049	8.6049
	Higher-order theory (Lim et al., 1998a,b,c)	1.9317	4.6088	4.6088	6.5233	6.5233	7.0731	8.6189	8.6189
	3-D exact solutions (Srinivas et al., 1970)	1.9342	4.6222	4.6222	–	–	7.1030	8.6617	8.6617
	3-D Ritz solutions (Leissa and Zhang, 1983)	1.9342	4.6222	4.6222	6.5234	6.5234	7.1030	8.6617	8.6617
	3-D DQ solutions (Malik and Bert, 1998)	1.9342	4.6250	4.6250	6.5234	6.5234	7.1064	8.6932	8.6932
	Present 3-D solutions	1.9342	4.6222	4.6222	6.5234	6.5234	7.1030	8.6617	8.6617
0.2	Mindlin theory (Mindlin et al., 1956)	1.7659	–	–	3.8576	3.8576	–	5.5729	6.5809
	3-D exact solutions (Srinivas et al., 1970)	1.7557	–	–	3.8991	3.8991	4.6128	5.6527	–
	3-D Ritz solutions (Leissa and Zhang, 1983)	1.7558	3.2617	3.2617	3.8991	3.8991	4.6128	5.6524	6.5234
	3-D DQ solutions (Malik and Bert, 1998)	1.7558	3.2617	3.2617	3.8999	3.8999	4.6127	5.6533	6.5236
	Present 3-D solutions	1.7558	3.2617	3.2617	3.8991	3.8991	4.6128	5.6524	6.5234
0.5	Higher-order theory (Lim et al., 1998a,b,c)	1.2451	1.3047	1.3047	1.8451	2.3079	2.3079	2.6094	2.6094
	3-D exact solutions (Srinivas et al., 1970)	1.2590	–	–	1.8451	–	–	–	–
	3-D Ritz solutions (Leissa and Zhang, 1983)	1.2590	1.3047	1.3047	1.8451	2.3312	2.3312	2.6094	2.6094
	Present 3-D solutions	1.2590	1.3047	1.3047	1.8451	2.3312	2.3312	2.6094	2.6094

plate with the thickness-side ratio $\gamma = 0.001$ is actually a stringent test case for 3-D vibration analysis. This case was solved using $13 \times 13 \times 3$ terms of admissible functions.

4. Numerical results

In this section, the proposed 3-D Ritz formulation is applied to investigate the free vibration of rectangular plates with four clamped edges. Three different aspect ratios $a/b = 1.0$; 1.5; 2.0 and nine different

Table 7
Rectangular plates with four clamped edges: the first 12 frequency parameters Δ for the AAA mode

a/b	t/b	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
1.0	0.1	8.8943	17.261	17.418	24.101	28.084	28.186	33.488	33.622	39.994	40.063	41.573	44.425
	0.2	6.4152	11.294	11.383	15.100	17.133	17.146	17.937	18.895	20.054	20.070	21.069	21.326
	0.3	4.8478	8.2132	8.2215	8.7918	9.5887	10.784	11.461	11.660	12.076	12.229	12.626	13.841
	0.4	3.8570	5.5137	6.0598	6.4370	6.5764	7.8148	7.9542	8.3465	8.6294	9.3686	9.3772	9.5791
	0.5	3.1918	3.9554	4.4868	5.2705	5.3190	5.9382	6.0465	6.5991	6.7743	7.0482	7.4649	7.6149
1.5	0.1	6.8456	10.924	15.910	16.877	19.021	23.830	23.954	27.049	29.507	29.922	31.643	33.427
	0.2	5.1303	7.7435	10.511	11.227	12.350	15.043	15.104	16.538	17.533	17.923	18.154	18.286
	0.3	3.9510	5.8168	7.6403	8.1925	8.4206	8.9264	9.0018	9.6942	10.718	10.766	11.033	11.427
	0.4	3.1746	4.6152	5.1625	5.6170	6.0637	6.3111	6.4375	6.9620	7.3547	7.7731	7.9235	8.3143
	0.5	2.6411	3.6441	3.8040	4.0795	4.6358	4.9639	5.2507	5.4895	5.7218	5.8958	6.0275	6.4018
2.0	0.1	6.1753	8.4480	12.063	15.432	16.641	17.171	19.988	21.869	23.725	26.676	27.531	28.055
	0.2	4.6734	6.2182	8.4875	10.221	11.165	11.286	12.931	14.071	15.018	16.325	17.079	17.125
	0.3	3.6125	4.7582	63560	7.4300	8.1771	8.1991	8.3126	8.7190	9.0213	9.3469	9.9351	10.127
	0.4	2.9076	3.8143	5.0156	5.0570	5.4257	5.6613	5.9393	6.3780	6.4203	6.5744	6.8284	7.2817
	0.5	2.4211	3.1697	3.5399	3.8759	4.0973	4.1706	4.7687	4.8905	5.0307	5.2565	5.3124	5.8538

Table 8
Rectangular plates with four clamped edges: the first 12 frequency parameters Δ for the AAS mode

a/b	t/b	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
1.0	0.1	14.870	18.248	26.671	27.378	30.661	36.434	36.704	39.851	39.992	41.918	44.491	48.492
	0.2	7.4369	9.1323	13.338	13.679	15.317	17.815	18.208	19.912	19.953	20.604	22.034	23.214
	0.3	4.9591	6.0847	8.8924	9.0962	10.155	10.788	11.643	12.645	12.994	13.016	13.308	13.517
	0.4	3.7200	4.5534	6.6691	6.6719	7.0594	7.4614	8.1010	8.1187	8.9616	9.0949	9.4532	9.4775
	0.5	2.9764	3.6275	4.8151	5.3348	5.5114	5.7480	5.8885	6.0125	6.3132	6.6670	7.0874	7.4504
1.5	0.1	12.654	15.477	19.195	24.380	26.496	27.192	30.025	31.619	35.198	35.469	37.966	39.157
	0.2	6.3317	7.7445	9.5977	12.175	13.242	13.594	14.943	15.772	17.310	17.714	18.762	19.153
	0.3	4.2234	5.1628	6.3962	8.0677	8.8143	9.0568	9.7692	10.347	10.810	11.255	11.810	11.895
	0.4	3.1688	3.8685	4.7923	5.9273	6.5203	6.7573	6.7981	7.0788	7.4934	7.8409	7.9279	8.1577
	0.5	2.5358	3.0888	3.8240	4.4439	4.8258	5.1794	5.3240	5.4597	5.5756	5.7936	5.8585	6.1055
2.0	0.1	11.801	14.298	15.807	20.326	21.688	26.249	26.706	27.634	28.210	31.934	32.733	33.635
	0.2	5.9082	7.1522	7.9069	10.159	10.838	13.122	13.346	13.809	14.068	15.934	16.189	16.774
	0.3	3.9425	4.7684	5.2710	6.7575	7.2067	8.7419	8.8700	9.1783	9.2857	10.211	10.409	10.737
	0.4	2.9588	3.5751	3.9509	5.0372	5.3721	6.4558	6.4833	6.6339	6.7972	6.8536	7.2312	7.3179
	0.5	2.3681	2.8579	3.1567	3.9593	4.2465	4.5964	4.7614	5.0947	5.2673	5.3795	5.4566	5.5947

Table 9
Rectangular plates with four clamped edges: the first 12 frequency parameters Δ for the ASA mode

a/b	t/b	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
1.0	0.1	6.3389	12.695	15.484	20.281	22.548	26.697	28.583	30.458	33.989	37.300	38.853	38.903
	0.2	4.7696	8.6944	10.242	13.009	14.193	16.329	17.475	17.646	18.412	19.336	20.086	20.869
	0.3	3.6693	6.4306	7.4360	8.4982	9.3728	10.001	10.157	11.135	11.669	12.283	12.470	12.970
	0.4	2.9403	5.0315	5.1989	5.9299	6.5907	7.2505	7.4646	7.9476	8.5000	8.8718	9.1616	9.3100
	0.5	2.4389	3.6541	4.1836	4.8226	4.9139	5.5814	5.9639	6.3963	6.4659	6.7606	7.1618	7.3133
1.5	0.1	3.8215	8.4956	11.008	14.548	14.922	19.932	21.311	22.314	24.066	26.521	28.405	30.269
	0.2	3.1127	6.2532	7.6838	9.8539	10.102	12.929	13.501	14.212	15.074	16.492	17.057	17.447
	0.3	2.5185	4.7770	5.7309	7.2612	7.4144	8.0017	9.3502	9.5918	9.8274	10.049	10.280	10.730
	0.4	2.0785	3.8215	4.5230	4.7970	5.7079	5.8478	6.3835	6.6315	7.1770	7.3205	7.3769	7.6628
	0.5	1.7552	3.1281	3.3293	3.7593	4.6695	4.7017	4.8010	4.9460	5.3684	5.5829	5.9527	6.1599
2.0	0.1	2.9688	5.6437	9.6946	10.451	12.422	14.630	15.609	19.758	20.134	20.892	22.435	24.605
	0.2	2.4821	4.4315	7.0704	7.3306	8.5899	10.025	10.517	12.888	13.122	13.259	14.161	15.537
	0.3	2.0475	3.5088	5.3803	5.4792	6.3877	7.4021	7.7282	7.8162	9.1569	9.3387	9.5097	9.5652
	0.4	1.7124	2.8649	4.2922	4.3326	4.6360	5.0461	5.7663	5.9761	6.0683	6.3842	6.7248	6.9331
	0.5	1.4593	2.4062	3.1193	3.5632	3.6101	4.1375	4.2893	4.7133	4.8082	4.9581	5.0119	5.1910

Table 10
Rectangular plates with four clamped edges: the first 12 frequency parameters Δ for the ASS mode

a/b	t/b	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
1.0	0.1	12.518	20.695	25.026	30.126	33.190	33.723	36.437	40.386	43.056	43.892	46.091	48.164
	0.2	6.2722	10.347	12.502	15.003	16.430	16.787	18.034	19.994	21.126	21.509	22.795	23.184
	0.3	4.1871	6.8953	8.2895	9.8007	10.422	11.104	11.546	11.983	12.639	12.888	13.393	13.593
	0.4	3.1425	5.1645	6.1051	6.6470	7.2777	7.6737	7.9083	8.2749	8.5055	8.8403	8.9307	9.1656
	0.5	2.5145	4.1096	4.5195	4.8977	5.4006	5.5063	6.0697	6.2325	6.3937	6.6995	6.7274	6.7770
1.5	0.1	9.6411	19.576	20.858	24.161	25.023	28.929	31.576	32.977	35.463	36.179	37.481	38.897
	0.2	4.8269	9.7911	10.424	12.003	12.491	14.424	15.710	16.481	17.588	17.904	18.499	19.144
	0.3	3.2210	6.5256	6.9317	7.8605	8.2727	9.5098	10.227	10.761	10.954	11.481	11.726	11.960
	0.4	2.4173	4.8889	5.1558	5.6427	6.0866	6.8330	6.8611	7.3885	7.4826	7.6914	7.7622	8.1861
	0.5	1.9345	3.9008	3.9683	4.2511	4.6249	4.9010	5.1983	5.2761	5.5410	5.6087	5.8211	5.9534
2.0	0.1	8.4139	16.730	19.647	21.400	22.612	24.872	27.609	28.616	30.150	32.798	33.188	34.449
	0.2	4.2104	8.3625	9.8198	10.682	11.297	12.398	13.756	14.130	15.037	16.390	16.485	17.170
	0.3	2.8087	5.5607	6.5401	7.0883	7.5001	8.1834	9.0243	9.0850	9.9231	10.622	10.899	10.914
	0.4	2.1074	4.1452	4.8918	5.2515	5.5616	5.9313	6.2171	6.5362	6.9608	7.1655	7.3042	7.5142
	0.5	1.6864	3.2735	3.8681	4.0724	4.2686	4.3803	4.7099	4.8552	4.9372	5.1035	5.3593	5.4383

thickness-side ratios varying from $t/b = 0.1$ to $t/b = 0.5$ with an increment of 0.1 are considered. Tables 7–14 show the first 12 frequency parameters Δ of different mode categories. In the computation, $11 \times 11 \times 6$ terms of admissible functions for each displacement function were used. It is shown that for plates with the same thickness-side ratio t/b, the natural frequency decreases with increase in the aspect ratio a/b. Moreover, observing and comparing the results in these tables, one can find that for thick plates, the natural frequencies of the symmetric modes in the thickness direction are close to those of the antisymmetric modes in the same direction. However for thin plates, the natural frequencies of the symmetric modes in the thickness direction are clearly higher than those of the antisymmetric modes. The same

Table 11
Rectangular plates with four clamped edges: the first 12 frequency parameters Δ for the SAA mode

a/b	t/b	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
1.0	0.1	6.3389	12.695	15.484	20.281	22.548	26.697	28.583	30.458	33.989	37.300	38.853	38.903
	0.2	4.7696	8.6944	10.242	13.009	14.193	16329	17.475	17.646	18.412	19.336	20.086	20.869
	0.3	3.6693	6.4306	7.4360	8.4982	9.3728	10.001	10.157	11.135	11.669	12.283	12.470	12.970
	0.4	2.9403	5.0315	5.1989	5.9299	6.5907	7.2505	7.4646	7.9476	8.5000	8.8718	9.1616	9.3100
	0.5	2.4389	3.6541	4.1836	4.8226	4.9139	5.5814	5.9639	6.3963	6.4659	6.7606	7.1618	7.3133
1.5	0.1	5.7835	8.5918	13.721	15.118	17.223	20.300	21.253	26.421	26.756	27.725	28.089	31.303
	0.2	4.3926	6.2835	9.4132	10.023	11.301	13.133	13.613	16.173	16.611	17.095	17.130	17.487
	0.3	3.3910	4.7885	6.9690	7.2792	8.2044	8.3494	8.9329	9.4514	9.7946	9.9359	10.608	11.541
	0.4	2.7224	3.8289	5.0175	5.4903	5.6094	5.8358	6.3280	6.5633	7.0853	7.4001	7.6136	7.9812
	0.5	2.2611	3.1763	3.5376	4.0536	4.4957	4.7120	4.8025	5.2628	5.3419	5.9956	6.0764	6.1900
2.0	0.1	5.6157	7.1234	10.111	14.255	14.997	16.158	18.449	19.194	21.755	24.653	25.872	26.328
	0.2	4.2736	5.3324	7.2847	9.7901	9.9485	10.670	12.042	12.602	13.929	15.569	16.118	16.183
	0.3	3.2987	4.1109	5.5170	7.2243	7.2508	7.7601	8.2913	8.5668	8.7340	9.1129	9.3034	9.6136
	0.4	2.6470	3.3079	4.3969	4.9731	5.2724	5.6830	5.7781	5.9326	6.0891	6.2828	6.8205	7.1084
	0.5	2.1979	2.7555	3.4885	3.6434	3.7517	4.2926	4.5365	4.7094	4.7383	5.0314	5.3710	5.5243

Table 12
Rectangular plates with four clamped edges: the first 12 frequency parameters Δ for the SAS mode

a/b	t/b	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
1.0	0.1	12.518	20.695	25.026	30.126	33.190	33.723	36.437	40.386	43.056	43.892	46.091	48.164
	0.2	6.2722	10.347	12.502	15.003	16.430	16.787	18.034	19.994	21.126	21.509	22.795	23.184
	0.3	4.1871	6.8953	8.2895	9.8007	10.422	11.104	11.546	11.983	12.639	12.888	13.393	13.593
	0.4	3.1425	5.1645	6.1051	6.6470	7.2777	7.6737	7.9083	8.2749	8.5055	8.8403	8.9307	9.1656
	0.5	2.5145	4.1096	4.5195	4.8977	5.4006	5.5063	6.0697	6.2325	6.3937	6.6995	6.7274	6.7770
1.5	0.1	11.565	15.492	19.763	23.175	27.461	28.616	31.223	33.324	34.130	34.925	37.882	39.722
	0.2	5.7961	7.7483	9.8803	11.586	13.719	14.284	15.604	16.366	16.874	17.283	18.682	19.843
	0.3	3.8702	5.1661	6.5700	7.7180	9.1121	9.4401	10.173	10.368	10.623	11.097	11.756	12.006
	0.4	2.9052	3.8739	4.8955	5.7746	6.5427	6.6697	6.8533	7.2513	7.5265	7.6568	7.8021	7.9941
	0.5	2.3251	3.0975	3.8601	4.5542	4.5694	4.8311	5.2776	5.3761	5.6124	5.7420	5.9377	5.9953
2.0	0.1	11.233	13.369	17.073	18.622	23.604	24.809	26.905	29.332	30.241	30.490	33.271	34.269
	0.2	5.6304	6.6896	8.5379	9.3115	11.795	12.392	13.440	14.640	15.068	15.221	16.323	16.991
	0.3	3.7597	4.4619	5.6870	6.2022	7.8404	8.2244	8.9293	9.6584	9.8722	10.116	10.176	10.648
	0.4	2.8224	3.3471	4.2554	4.6414	5.8056	6.0932	6.4985	6.6255	6.8747	6.9243	7.2265	7.5232
	0.5	2.2589	2.6776	3.3880	3.6969	4.3968	4.5289	4.7822	4.8190	5.1471	5.3480	5.4090	5.4698

conclusions can also be drawn from Tables 3 and 4 for simply-supported plates. This means that for thin plates, a larger number of lower natural frequencies are confined to the antisymmetric modes in the thickness direction, i.e. AAA, ASA, SAA and SSA.

The effect of Poisson's ratio on natural frequencies of clamped square thick plates with a thickness-side ratio $t/b = 0.5$ has also been studied. Four different Poisson's ratios $v = 1/6$; 1/4; 1/3; 2/5 were considered. In Table 15, the first 12 frequency parameters Ω of different mode categories are tabulated with respect to the Poisson's ratios. It is shown that the natural frequency of antisymmetric modes in the thickness direction monotonically decreases with the increase in Poisson's ratio. However, the variation of natural frequency for the symmetric modes in the thickness direction does not display a monotonic trend.

Table 13
Rectangular plates with four clamped edges: the first 12 frequency parameters Δ for the SSA mode

a/b	t/b	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
1.0	0.1	3.3176	10.488	10.587	16.044	20.897	20.949	25.070	25.229	32.662	32.688	32.742	36.021
	0.2	2.7241	7.3208	7.4211	10.646	13.241	13.296	15.612	15.656	19.099	19.442	19.631	19.813
	0.3	2.2146	5.4591	5.5380	7.7764	9.4057	9.5296	10.185	10.774	11.130	11.194	11.269	12.544
	0.4	1.8321	4.3258	4.3857	6.0848	6.5653	7.0637	7.3753	7.5282	7.6821	8.4600	8.6153	8.7659
	0.5	1.5488	3.5768	3.6226	4.8775	4.9781	5.2762	5.6733	6.0855	6.1052	6.4907	6.9964	7.0213
1.5	0.1	2.5399	5.8704	10.099	11.547	12.512	17.055	18.526	20.608	22.468	23.113	26.056	26.242
	0.2	2.1459	4.5462	7.0947	8.1240	8.6268	11.310	12.137	13.088	14.171	14.669	16.177	16.301
	0.3	1.7803	3.5660	5.3015	6.0805	6.4032	8.2577	8.7091	9.0781	9.3827	10.085	10.153	10.488
	0.4	1.4931	2.8956	4.2048	4.8101	5.0493	5.6884	6.4511	6.6294	6.8490	6.9531	7.3384	7.5394
	0.5	1.2742	2.4256	3.4775	3.9102	4.1053	4.2169	4.9087	5.1412	5.2906	5.5717	5.6562	6.0134
2.0	0.1	2.3231	4.0954	7.5332	9.9594	11.272	12.077	13.877	17.323	17.583	20.503	21.535	22.112
	0.2	1.9735	3.3318	5.6969	7.0042	7.8617	8.5243	9.4844	11.561	11.660	13.026	13.638	14.188
	0.3	1.6410	2.6982	4.4167	5.2351	5.8657	6.3877	7.0168	83223	8.5065	8.5614	9.3203	9.6822
	0.4	1.3776	2.2318	3.5646	4.1530	4.6388	5.0143	5.2466	5.5288	6.3988	6.5497	6.6503	6.6678
	0.5	1.1764	1.8899	2.9718	3.4315	3.6279	3.8570	4.1963	4.5413	4.7426	4.8655	5.1482	5.3641

Table 14
Rectangular plates with four clamped edges: the first 12 frequency parameters Δ for the SSS mode

a/b	t/b	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}
1.0	0.1	20.624	23.562	25.579	30.648	35.170	37.067	39.788	44.472	44.804	46.475	47.369	49.158
	0.2	10.319	11.712	12.768	15.259	17.562	18.390	19.777	20.936	21.186	22.676	23.006	24.052
	0.3	6.8672	7.6761	8.4610	9.9877	11.434	11.538	11.943	11.983	12.698	13.077	13.094	13.346
	0.4	5.1259	5.5215	6.2378	7.0392	7.4154	7.4897	7.7435	8.1926	8.5382	8.7877	8.8474	9.2916
	0.5	4.0645	4.0736	4.7932	5.1273	5.2903	5.4721	5.7155	6.3347	6.3663	6.5609	6.9029	7.0701
1.5	0.1	15.395	20.937	22.487	25.032	28.052	30.210	32.771	34.260	34.998	38.972	39.652	41.821
	0.2	7.7010	10.454	11.222	12.492	13.965	14.914	16.335	17.075	17.400	19.301	19.631	20.604
	0.3	5.1271	6.9398	7.4267	8.2737	9.1328	9.5494	10.693	11.171	11.263	11.626	11.673	11.934
	0.4	3.8312	5.1412	5.4756	6.0972	6.2666	6.7644	7.2759	7.3375	7.4686	7.5632	7.8984	8.1424
	0.5	3.0427	3.9563	4.2320	4.4822	4.7213	4.9580	5.1164	5.2285	5.3873	5.7507	5.9840	6.0359
2.0	0.1	12.392	19.772	20.629	22.722	24.313	24.979	27.451	30.431	32.814	33.075	33.486	35.884
	0.2	6.2002	9.8870	10.305	11.314	12.091	12.459	13.688	15.133	16.267	16.380	16.710	17.751
	0.3	4.1320	6.5846	6.8434	7.4471	7.9399	8.2389	9.0398	9.8564	10.104	10.807	11.033	11.330
	0.4	3.0943	4.9227	5.0761	5.4083	5.7323	6.0499	6.4793	6.6955	7.0106	7.2861	7.3669	7.4345
	0.5	2.4682	3.8884	3.9131	4.0559	4.3534	4.5282	4.7823	4.9254	5.0799	5.1748	5.3086	5.3566

5. Conclusions

The free vibration characteristics of thick rectangular plates, based on the linear, small-strain 3-D elasticity theory, have been investigated. The Ritz method is applied to derive the eigenvalue equation. The spatial displacement components in the three coordinate directions are described by admissible functions comprising Chebyshev polynomials multiplied by the corresponding boundary functions. High accuracy, stable numerical computation and rapid convergence have been observed in the analysis. For demonstration, the natural frequencies of rectangular plates of different aspect ratios, thickness-side ratios and Poisson's ratios with four clamped edges have been studied in detail. The method is also applicable to very thin plates. The first 12 frequency parameters for various mode categories are given. These results can serve

Table 15
Square plate with four clamped edges: the effect of Poisson's ratio v on the frequency parameters

Modes	v	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9	Ω_{10}	Ω_{11}	Ω_{12}
AAA	1/6	4.9052	6.2253	6.9204	8.1049	8.2120	9.3422	9.5104	10.234	10.444	10.972	11.600	11.743
	1/4	4.8143	6.0200	6.7720	7.9540	8.0388	9.0368	9.2003	9.9870	10.234	10.681	11.315	11.506
	1/3	4.7371	5.8349	6.6602	7.8176	7.8823	8.7603	8.9215	9.7722	10.041	10.429	11.038	11.284
	2/5	4.6839	5.6996	6.6001	7.7163	7.7669	8.5574	8.7184	9.6194	9.8973	10.254	10.826	11.115
AAS	1/6	4.6785	5.4950	7.1361	8.3844	8.5961	8.9356	9.0257	9.4589	9.5654	10.181	10.862	11.081
	1/4	4.5276	5.4381	7.1507	8.1155	8.3507	8.7102	8.8691	9.1526	9.4814	10.000	10.696	11.202
	1/3	4.3927	5.4084	7.2277	7.8721	8.1647	8.5036	8.7538	8.8682	9.3855	9.9672	10.515	10.993
	2/5	4.2952	5.4027	7.3027	7.6934	8.0771	8.3538	8.6592	8.7054	9.2851	10.091	10.430	10.739
ASA (SAA)	1/6	3.7421	5.6429	6.4312	7.4386	7.7042	8.5925	9.1990	9.9116	10.028	10.502	11.267	11.297
	1/4	3.6760	5.5178	6.3119	7.2886	7.4651	8.4206	9.0108	9.6851	9.7850	10.236	10.902	11.054
	1/3	3.6218	5.4222	6.2072	7.1430	7.2615	8.2849	8.8393	9.4646	9.5757	10.009	10.563	10.836
	2/5	3.5861	5.3697	6.1319	7.0195	7.1305	8.2010	8.7116	9.2964	9.4267	9.8516	10.314	10.674
ASS (SAS)	1/6	3.7294	6.4507	6.6921	7.4798	8.0487	8.4259	9.3143	9.3562	9.8448	10.080	10.388	10.532
	1/4	3.7353	6.2492	6.7160	7.3591	8.0722	8.2745	9.1338	9.2793	9.6523	10.152	10.187	10.213
	1/3	3.7756	6.0658	6.7723	7.3022	8.0391	8.2055	9.0395	9.3582	9.4901	9.9167	9.9586	10.090
	2/5	3.8375	5.9319	6.7963	7.3311	7.9704	8.1647	9.0297	9.3387	9.4575	9.7547	9.9775	10.187
SSA	1/6	2.3733	5.4749	5.5607	7.4992	7.6556	8.0689	8.8348	9.3840	9.4353	10.018	10.785	10.858
	1/4	2.3330	5.3872	5.4624	7.3553	7.5121	7.9407	8.5968	9.1937	9.2306	9.8045	10.570	10.618
	1/3	2.3012	5.3132	5.3770	7.2417	7.3847	7.8446	8.3968	9.0198	9.0448	9.6249	10.370	10.400
	2/5	2.2817	5.2619	5.3163	7.1696	7.2917	7.7913	8.2650	8.8898	8.9072	9.5038	10.219	10.239
SSS	1/6	6.1500	6.2070	7.0981	7.3651	7.5861	7.9608	8.7051	9.3986	9.7643	9.9696	10.686	11.016
	1/4	6.0995	6.1248	7.2479	7.4324	7.7125	8.0188	8.5533	9.4600	9.5676	9.8359	10.443	10.727
	1/3	6.0496	6.0592	7.0935	7.7863	8.0721	8.3183	8.6043	9.3928	9.5395	9.8010	10.219	10.459
	2/5	6.0086	6.0192	6.9623	7.8935	8.5104	8.6610	8.9854	9.2719	9.6286	10.035	10.058	10.288

as benchmark solutions for validating approximate two-dimensional theories and new computational techniques in future.

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